$\mathcal{N}$-dimensional fractional diffusion equation and Green function approach: Spatially dependent diffusion coefficient and external force

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We investigate an $\mathcal{N}$-dimensional fractional diffusion equation with radial symmetry by using the Green function approach. We consider, in our analysis, the spatial dependence on the diffusion coefficient and the presence of an external force. In particular, we employ boundary conditions in a finite interval and after we extend it to a semi-infinite interval. We also show that a rich class of diffusive processes, including normal and anomalous ones, can be obtained from the solutions found here.

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I. INTRODUCTION

The fractional approach applied to anomalous diffusion has attracted the attention of many scientists [1–7]. In fact, the diffusion equations that emerge from this approach have been applied in a rich variety of scenarios such as relaxation to equilibrium in systems (such as polymer chains and membranes) with long temporal memory [8–11], anomalous transport in disordered systems [12], diffusion on fractals [13], and modeling of non-Markovian dynamical processes in protein folding [14]. Formal properties concerning the fractional diffusion equations have also been investigated. For instance, in [15] the fractional diffusion and wave equations are discussed, in [16] boundary value problems for fractional diffusion equations are studied, in [17] a fractional Fokker-Planck equation is derived from a generalized master equation, in [18] the behavior of fractional diffusion at the origin is analyzed, in [19–25] the solutions of the time fractional diffusion equation are obtained, in [26] a harmonic analysis of random fractional diffusion-wave equations is done, and in [27] a fractional Kramers equation is introduced. In this direction, we focus on the analysis of the following fractional diffusion equation:

$$\frac{\partial^{\gamma}}{\partial t^{\gamma}}\rho = \frac{1}{r^{\mathcal{N}-1}}\frac{\partial}{\partial r}\left\{ r^{\mathcal{N}-1}\left[\mathcal{D}(r)\frac{\partial}{\partial r}\rho - F(r)\rho \right] \right\} \tag{1}$$

where $F(r)$ is an external force, $0<\gamma\leqslant 1$, and $\mathcal{D}(r)$ is a diffusion coefficient with a spatial dependence. The time fractional derivative is considered in the Caputo representation [28]. We consider a spatial dependence on the diffusion coefficient, i.e., $\mathcal{D}(r)=\mathcal{D}r^{-\theta}$, and the presence of an external force $F(r)=-kr+\mathcal{K}r^{\alpha}$ with $\alpha=-1-\theta$. We investigate the solutions for Eq. (1) by using the Green function method taking finite and semi-infinite boundary conditions into account. It is interesting to note that these kinds of boundary conditions, which appear in several physical contexts, have not been properly investigated in the fractional context. Notice that Eq. (1) recovers the usual radial diffusion equation for $\gamma=1$. For Eq. (1), it can be verified that $\int_0^{\infty} dr\, r^{\mathcal{N}-1}\rho$ is time independent (hence, if ρ is normalized at $t=0$, it will remain so forever). Indeed, if we write the equation in the form $\partial_t\rho = -r^{1-\mathcal{N}}\partial_r(r^{\mathcal{N}-1}\mathcal{J})$ and assume the boundary conditions $\mathcal{J}(\infty,t)\rightarrow 0$, it can be shown that $\int_0^{\infty} dr\, r^{\mathcal{N}-1}\rho$ is a constant of motion.

The plan of this work is to investigate the time dependent solutions of Eq. (1) by considering several situations, such as the absence of external forces, taking a spatial time dependence in the diffusion coefficient into account, and introducing an external force in the system. In this context, we first study the situations characterized by boundary conditions defined in a finite interval and then we extend our analysis to a semi-infinite interval. These developments are done in Sec. II. In Sec. III, we present the conclusions.

II. FRACTIONAL DIFFUSION EQUATION

We start our discussion by considering an $\mathcal{N}$-dimensional fractional diffusion equation with radial symmetry in the absence of external forces, taking a spatial dependence on the diffusion coefficient, i.e., $\mathcal{D}(r)=\mathcal{D}r^{-\theta}$, into account. This spatial dependence on the diffusion coefficient has been used to investigate several physical situations such as the fast electrons in a hot plasma in the presence of an electric field [29], turbulence [30,31], and diffusion on fractals [32,33]. For this case, Eq. (1) is given by

$$\frac{\partial^{\gamma}}{\partial t^{\gamma}}\rho = \frac{\mathcal{D}}{r^{\mathcal{N}-1}}\frac{\partial}{\partial r}\left\{ r^{\mathcal{N}-1}r^{-\theta}\frac{\partial}{\partial r}\rho \right\}. \tag{2}$$

It is interesting to note that for $\gamma=1$ and $\theta=0$ Eq. (2) recovers the usual case and for $\theta=0$ it recalls a diffusion equation used in [18] to investigate the behavior of the solution near the origin when the free boundary condition is employed. Here, we analyze Eq. (2) by using the boundary condition $\rho(a,t)=0$. Similar boundary condition is found in the analysis of polymer dynamics, stratified porous media, and photoconductivity in amorphous semiconductors. By solving the above equation subjected to this boundary condition, we obtain

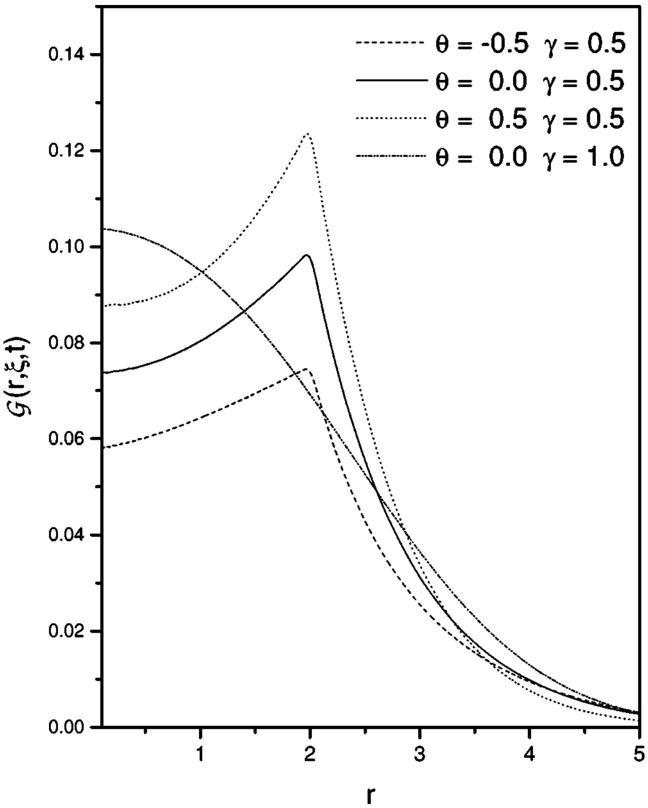

FIG. 1. Behavior of $\mathcal{G}(r,\xi,t)$ versus r for typical values of θ and γ, by considering, for simplicity, $t=1.0$, $\xi=2.0$, $a=6.0$, $\mathcal{N}=3.0$, and $\mathcal{D}=1.0$, in arbitrary units.

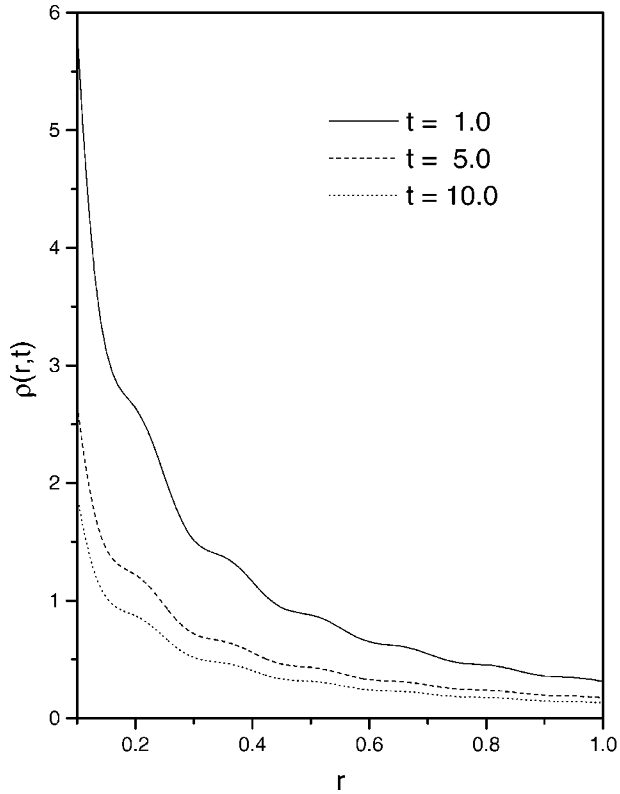

FIG. 2. Behavior of $\rho(r,t)$ versus r for typical values of t by considering, for simplicity, $\gamma=0.5$, $a=6.0$, $\mathcal{N}=3.0$, $\theta=0$, and $\mathcal{D}=1.0$. The initial condition used for this case is the $\mathcal{N}$-dimensional Dirac δ function, i.e., $\rho(r,0)=\delta(r)/r^{\mathcal{N}-1}$, in arbitrary units.

$$\rho(r,t)=\int_0^a d\xi\,\xi^{\mathcal{N}-1}\tilde{\rho}(\xi)\mathcal{G}(r,\xi,t),$$

$$\mathcal{G}(r,\xi,t)=\frac{2+\theta}{a^{2+\theta}}\sum_{n=0}^{\infty}\frac{\xi^{(2+\theta-\mathcal{N})/2}r^{(2+\theta-\mathcal{N})/2}}{\left\{J_{\mathcal{N}/(2+\theta)}\left(\frac{2\lambda_n}{2+\theta}a^{(2+\theta)/2}\right)\right\}^2}E_\gamma(-\mathcal{D}\lambda_n^2t^\gamma)$$
$$\times J_{(\mathcal{N}-2-\theta)/(2+\theta)}\left(\frac{2\lambda_n\xi^{(2+\theta)/2}}{2+\theta}\right)$$
$$\times J_{(\mathcal{N}-2-\theta)/(2+\theta)}\left(\frac{2\lambda_n r^{(2+\theta)/2}}{2+\theta}\right), \tag{3}$$

where $\mathcal{N}\geqslant 2+\theta$, λ_n (eigenvalue) is obtained from the equation $J_{(\mathcal{N}-2-\theta)/(2+\theta)}\{[2\lambda_n/(2+\theta)]a^{(2+\theta)/2}\}=0$ and $\rho(r,0)=\tilde{\rho}(r)$ is the initial condition. In Eq. (3), $\mathcal{G}(r,\xi,t)$ is the Green function and $E_\gamma(x)$ is the Mittag-Leffler function which is given by $E_\gamma(x)=\sum_{n=0}^{\infty}x^n/\Gamma(1+\gamma n)$. The Mittag-Leffler function is an extension of the usual exponential and the presence of this function in Eq. (3) is a consequence of the changes produced in the waiting time probability density function by the fractional derivative. In order to illustrate the effect produced due to the time fractional derivative and the spatial dependence of the diffusion coefficient, we plot the Green function in Fig. 1. In Fig. 2, we show the time evolution behavior of Eq. (3) by employing, for simplicity, $\theta=0$ and the initial condition $\rho(r,0)=\delta(r)/r^{\mathcal{N}-1}$.

We may extend the above result found for Eq. (2) by considering $a\to\infty$. To obtain this extension, it is useful to use

$$\rho(r,t)=\int_0^\infty dk\,\mathcal{C}(k,t)\Psi(r,k),$$

$$\Psi(r,k)=r^{(2+\theta-\mathcal{N})/2}J_{(\mathcal{N}-2-\theta)/(2+\theta)}\left(\frac{2kr^{(2+\theta)/2}}{2+\theta}\right), \tag{4}$$

where $\mathcal{C}(k,t)$ is the kernel to be found. By substituting Eq. (4) in Eq. (2), we obtain

$$\frac{d^\gamma}{dt^\gamma}\mathcal{C}(k,t)=-\mathcal{D}k^2\mathcal{C}(k,t). \tag{5}$$

By solving Eq. (5), we found $\mathcal{C}(k,t)=\mathcal{C}(k,0)E_\gamma(-k^2\mathcal{D}t^\gamma)$, where $\mathcal{C}(k,0)$ is determined by the initial condition. By using the initial condition $\rho(r,0)=\tilde{\rho}(r)$, we verify

$$\mathcal{C}(k,0)=\frac{2k}{2+\theta}\int_0^\infty d\xi\,\xi^{\mathcal{N}-1}\tilde{\rho}(\xi)\Psi(\xi,k). \tag{6}$$

Thus, the solution for this case is given by

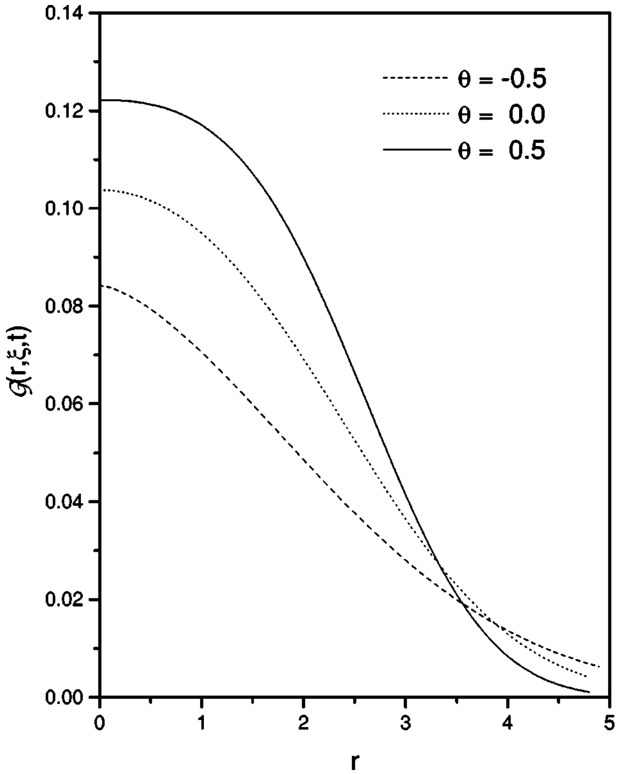

FIG. 3. Behavior of $\mathcal{G}(r,\xi,t)$ versus r for a typical values of θ by considering, for simplicity, $t=1.0$, $\xi=2.0$, $\mathcal{N}=3.0$, and $\mathcal{D}=1.0$, in arbitrary units.

$$\rho(r,t) = \int_0^\infty d\xi\, \xi^{\mathcal{N}-1}\tilde{\rho}(\xi)\mathcal{G}(r,\xi,t),$$

$$\mathcal{G}(r,\xi,t) = \frac{2}{2+\theta}\int_0^\infty dk\, k\Psi(\xi,k)\Psi(r,k)E_\gamma(-k^2\mathcal{D}t^\gamma). \quad (7)$$

In particular, for $\gamma=1$, we can simplify the above equation by using the identity

$$\int_0^\infty dk\, kJ_\nu(\alpha k)J_\nu(\beta k)e^{-a^2k^2} = \frac{1}{2a^2}e^{-(\beta^2+\alpha^2)/4a^2}I_\nu\left(\frac{\alpha\beta}{2a^2}\right) \quad (8)$$

to obtain

$$\mathcal{G}(r,\xi,t) = \frac{e^{-(r^{2+\theta}+\xi^{2+\theta})/(2+\theta)^2\mathcal{D}t}/(\mathcal{D}t)}{(2+\theta)(\xi r)^{(\mathcal{N}-2-\theta)/2}} \times I_{(\mathcal{N}-2-\theta)/(2+\theta)}\left[\frac{2(\xi r)^{(2+\theta)/2}}{(2+\theta)^2\mathcal{D}t}\right], \quad (9)$$

where $I_\nu(x)$ is a modified Bessel function (see Fig. 3). In this context, an interesting result emerges from Eq. (7) for $\gamma=1$ with the initial condition $\rho(r,0)=\delta(r)/r^{\mathcal{N}-1}$. In fact, for this case Eq. (7) is reduced to $\rho(r,t)\propto e^{-r^{2+\theta}/(2+\theta)^2\mathcal{D}t}/t^{\mathcal{N}/(2+\theta)}$ by using Eq. (9) and this initial condition. This distribution has been applied to investigate situations related to turbulence [30] and diffusion on fractals [32,33]. In particular, in [33] the diffusion equations proposed in the literature to investigate diffusion on fractals are reviewed and critically discussed. Note also that the asymptotic expression for Eq. (9) taking the large argument for $I_\nu(x)$ into account is given by

$$\mathcal{G}(r,\xi,t) \sim \frac{(\xi r)^{(2+\theta-2\mathcal{N})/4}}{\sqrt{4\pi\mathcal{D}t}}e^{-(r^{(2+\theta)/2}-\xi^{(2+\theta)/2})^2/(2+\theta)^2\mathcal{D}t}. \quad (10)$$

The above equation can be considered as an extension of the asymptotic results reported in [2] for homogeneous and isotropic random walk models. The asymptotic behavior for the second moment associated with this process is $\langle r^2\rangle \sim t^{2/(2+\theta)}$ for long times. In particular, we can verify from this asymptotic behavior for the second moment that $0<\theta$, $\theta=0$, and $-2<\theta<0$ correspond to sub-, normal, and superdiffusive cases, respectively.

Let us incorporate an external force in Eq. (2). More precisely, we consider the external force $F(r)=-kr+\mathcal{K}r^\alpha$ with $\alpha=-1-\theta$, the boundary condition $\rho(\infty,t)=0$, and the initial condition $\rho(r,0)=\tilde{\rho}(r)$. This external force leads us to an extension of the Ornstein-Uhlenbeck process [34] and the Rayleigh process [35]. In addition, it is also similar to the one used in [36] to investigate new solutions for the nonlinear diffusion equation. Notice that to obtain an exact solution for Eq. (2) in the presence of the above external force with a generic α is a hard task. For this reason and to make it possible to obtain an analytical solution in a closed form, we have considered the relation between α and θ given by $\alpha=-1-\theta$. In order to obtain the solution, we expand $\rho(r,t)$ in terms of the eigenfunctions, i.e., we employ

$$\rho(r,t) = r^{\mathcal{K}/\mathcal{D}}e^{-kr^{2+\theta}/(2+\theta)\mathcal{D}}\sum_{n=0}^\infty \Psi_n(r)\Phi_n(t) \quad (11)$$

with $\Psi_n(r)$ (eigenfunction) determined by the spatial equation and $\Phi_n(t)$ obtained from the time equation. After some calculation, it is possible to show that

$$\Psi_n(r) = L_n^{(\overline{\alpha})}\left(\frac{kr^{2+\theta}}{(2+\theta)\mathcal{D}}\right),$$

$$\Phi_n(t) = \frac{(2+\theta)\Gamma(n+1)}{\Gamma((\mathcal{K}+\mathcal{N}\mathcal{D})/(2+\theta)\mathcal{D}+n)}\left(\frac{k}{(2+\theta)\mathcal{D}}\right)^{(\mathcal{K}+\mathcal{N}\mathcal{D})/(2+\theta)\mathcal{D}} \times E_\gamma(-\lambda_n t^\gamma)\int_0^\infty d\xi\, \xi^{\mathcal{N}-1}\tilde{\rho}(\xi)L_n^{(\overline{\alpha})}\left(\frac{k\xi^{2+\theta}}{(2+\theta)\mathcal{D}}\right) \quad (12)$$

with $\overline{\alpha}=\{(\mathcal{K}+\mathcal{N}\mathcal{D})/[(2+\theta)\mathcal{D}]\}-1$, where $L_n^{(\overline{\alpha})}(x)$ are associated Laguerre polynomials and $\lambda_n=(2+\theta)nk$. This result extends the result found in [2] for a linear external force and for $\gamma=0$, $\mathcal{N}=1$, and $\theta=0$ we recover the solution for the Rayleigh process present in [35]. It is also interesting to note that for this case the stationary solution is equal to the usual one. In particular, the second moment for this case considering, for simplicity, $\theta=0$, is given by $\langle r^2\rangle=2(\mathcal{N}\mathcal{D}+\mathcal{K})t^\gamma E_{\gamma,\gamma+1}(-2kt^\gamma)$, where $E_{\mu,\beta}(x)=\sum_{n=0}^\infty x^n/\Gamma(\mu n+\beta)$ is the generalized Mittag-Leffler function.

III. SUMMARY AND CONCLUSIONS

In summary, we have investigated the solutions for an $\mathcal{N}$-dimensional fractional diffusion equation within radial symmetry. We have obtained the solution for this equation by considering the absence of an external force and taking finite and semi-infinite boundary conditions into account. We have also considered the presence of an external force. This result is in agreement with the results found in [2]. For time dependent solutions, we have the presence of the Mittag-Leffler function, which is an extension of the usual exponential, in both cases (i.e., for the free case and in the presence of external forces). In particular, the presence of this function in the solutions is a consequence of the changes produced in the waiting time probability density function by the fractional derivative. Finally, we expect that the results presented here can be useful in the investigation of systems that exhibit anomalous diffusion.

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